Distinct scalings for mean first-passage time of random walks on scale-free networks with the same degree sequence

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In general, the power-law degree distribution has profound influence on various dynamical processes defined on scale-free networks. In this paper, we will show that power-law degree distribution alone does not suffice to characterize the behavior of trapping problems on scale-free networks, which is an integral major theme of interest for random walks in the presence of an immobile perfect absorber. In order to achieve this goal, we study random walks on a family of one-parameter (denoted by q) scale-free networks with identical degree sequence for the full range of parameter q, in which a trap is located at a fixed site. We obtain analytically or numerically the mean first-passage time (MFPT) for the trapping issue. In the limit of large network order (number of nodes), for the whole class of networks, the MFPT increases asymptotically as a power-law function of network order with the exponent obviously different for different parameter q, which suggests that power-law degree distribution itself is not sufficient to characterize the scaling behavior of MFPT for random walks at least trapping problem, performed on scale-free networks.

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I. INTRODUCTION

As a fundamental stochastic process, random walks have received considerable attention from the scientific society since they found a wide range of distinct applications in various theoretical and applied fields, such as physics, chemistry, biology, and computer science, among others [1–3]. Among a plethora of interesting issues of random walks, trapping is an integral major one, which plays an important role in an increasing number of disciplines. The so-called trapping issue that was first introduced in [4] is a random-walk problem, where a trap is positioned at a fixed location, absorbing all particles that visit it. The highly desirable quantity closely related to the trapping issue is the first-passage time (FPT) also called trapping time (TT). The FPT for a given site (node and vertex) is the time spent by a walker starting from the site to hit the trap node for the first time. This quantity is very important since it underlies many physical processes [5,6]. The average of first-passage times over all starting nodes is referred to as the mean first-passage time (MFPT) or mean trapping time (MTT), which is frequently used to measure the efficiency of the trapping problem.

One of the most important questions in the research of trapping is determining its efficiency, namely, showing the dependence relation of MFPT on the size of the system where the random walks are performed. Previous studies have provided the answers to the corresponding problems in some particular graphs with simple structure, such as regular lattices [4], Sierpinski fractals [7,8], T-fractal [9], and so forth. However, recent empirical studies [10–12] uncovered that many (perhaps most) real networks are scale-free characterized by a power-law degree distribution $P(k) \sim k^{-\gamma}$ with the exponent γ belonging to interval [2,3], which cannot be described by above simple graphs [13]. Thus, it appears quite natural and important to explore the trapping issue on scale-free networks. In recent work [14–16], we have shown that scale-free property may substantially improve the efficiency of the trapping problem: the MFPT behaves linearly or sublinearly with the order (number of nodes) of the scale-free networks, which is in sharp contrast to the superlinear scaling obtained for above-mentioned simple graphs [4,7–9]. It was speculated that the high efficiency of trapping on scale-free networks is attributed to their power-law property. Although scale-free feature can strongly affect the various dynamics occurring on networks, it was shown that the power-law degree distribution (even degree sequence) itself does not suffice to characterize some dynamical processes on scale-free networks, e.g., synchronization [17,18], disease spreading [19,20], and the like. Thus far, it is still not known whether degree sequence is sufficient to characterize the behavior of trapping problem on scale-free networks although it has been shown that the exponent γ of power-law degree distribution does not suffice [14,15,21,22].

In this paper, we study the trapping problem on a class of scale-free networks with the same degree sequence, which are dominated by a tunable parameter q [23]. We determine

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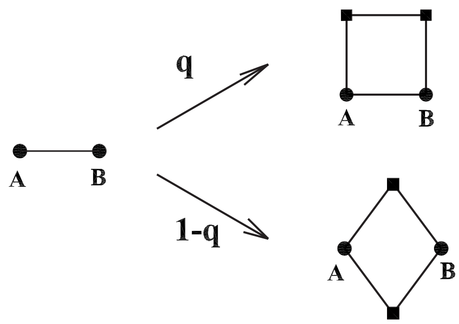

FIG. 1. Iterative method of the network construction. Each edge is replaced by either of the connected clusters on the right-hand side of arrows with a certain probability, where black squares represent new nodes.

separately the explicit formulas of the mean first-passage time for the two limiting cases of $q=1$ and $q=0$. We show that in both cases the MFPT increases as a power-law function of the network order, with the exponent less than 1 for $q=1$ and equal to 1 for $q=0$. We also study numerically the MFPT for the case of $0<q<1$, finding that it is also a power-law function of network order with the exponent $\theta(q)$ depending on parameter q. We demonstrate that in the full range of $0\le q\le 1$, $\theta(q)$ is a decreasing function of q, which belongs to the interval $[\ln 3/\ln 4, 1]$. Our findings indicate that the power-law degree distribution by itself is not sufficient to characterize the trapping process taking place on scale-free networks.

II. SCALE-FREE NETWORKS WITH IDENTICAL DEGREE SEQUENCE

The networks in question are built iteratively [23], see Fig. 1. We represent by $H_n(n\ge 0)$ the networks after n iterations (the number of iterations is also called generation hereafter). Then the networks are constructed as follows. For $n=0$, the initial network H_0 consists of two nodes connected to each other by an edge (a link). For $n\ge 1$, H_n is obtained from H_{n-1}. That is to say, to obtain H_n, one can replace each link existing in H_{n-1} either by a connected cluster of links in the top right of Fig. 1 with probability q, or by the connected cluster on the bottom right with complementary probability $1-q$. Repeat the growth process n times, with the graphs obtained in the limit $n\to\infty$. In Figs. 2 and 3, we present the growing processes of two special networks corresponding to $q=0$ and $q=1$, respectively.

Let $L(n)$ be the number of nodes created at generation n and E_n be the total number of all edges present at generation n. By construction, we have $E_n=4E_{n-1}$. Considering the initial condition $E_0=2$, it leads to $E_n=4^n$. Since each existing edge at a given generation will create two new nodes at the next generation, then, at each generation n_i ($n_i\ge 1$) the number of newly introduced nodes is $L(n_i)=2E_{n_i-1}=2\times 4^{n_i-1}$. Thus, at generation n the network order is

$$V_n=\sum_{n_i=0}^{n} L(n_i)=\frac{2}{3}(4^n+2). \quad (1)$$

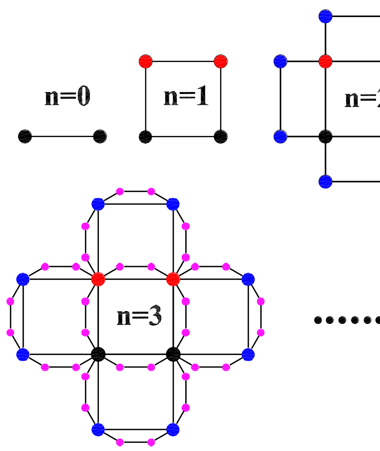

FIG. 2. (Color online) Illustration of the first several iterations of the network for a particular case $q=1$.

Let $k_i(n)$ be the degree of a node i at generation n, which was created at generation n_i ($n_i>0$). Then,

$$k_i(n)=2^{n-n_i+1}. \quad (2)$$

Note that the two nodes created at generation 0 have the same degree as that of the nodes added at generation 1. From Eq. (2), it is obvious that after each new iteration the degree of a node doubles, i.e.,

$$k_i(n)=2k_i(n-1). \quad (3)$$

The networks considered exhibit some interesting topological properties. Their nodes have same degree sequence (thus the same degree distribution), independent of the value of parameter q. Concretely, the networks have a power-law degree distribution $P(k)\sim k^{-\gamma}$ with the exponent $\gamma=3$ [23]. On the other hand, since there is no triangle in the whole class of the networks, the clustering coefficient is zero. Although the degree distribution and clustering coefficient do not depend on parameter q, other structural characteristics are closely related to q. For example, for $q=1$, the network is reduced to the (1, 3)-flower introduced in [24]. In this case, it is a small world, its average path length (APL), defined as the mean of shortest distances between all pairs of nodes, grows logarithmically with the network order [23]; at the same time, it is a nonfractal network [25,26]. While for $q=0$, it corresponds to the (2, 2)-flower addressed in [24] that

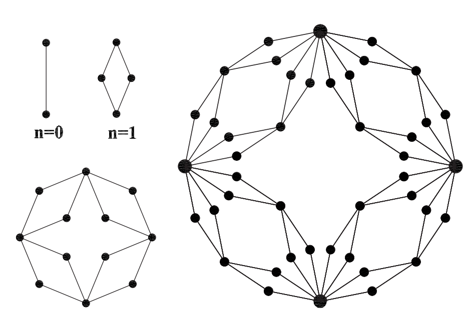

FIG. 3. Sketch of the iteration process of the network for the limiting case of $q=0$.

is exactly the hierarchical lattice proposed by Berker and Ostlund [27] and has been extensively studied by many authors [28–33]. For this case, the network is not small world with the APL increasing as a square power of the network order [30,31]; moreover, it is fractal with the fractal dimension $d_B=2$ [31]. When q increases from 0 to 1, the networks undergo a transition from fractal to nonfractal scalings and exhibit a crossover from "large" to small worlds at the same time [23]; these similar phenomena are also observed in a family of treelike networks [34].

The peculiar topological features make the networks unique within the category of scale-free networks since these particular structures strongly affect the dynamical processes defined on the networks. For instance, different thresholds of bond percolation were recently observed in the networks, which implies that power-law degree distribution (even degree sequence) alone does not suffice to characterize the percolation threshold on scale-free networks under bond percolation [23,32]. In what follows, we will study random walks with a single immobile trap on the networks. We will show that the degree sequence and thus the degree distribution are not sufficient to determine the scalings for MFPT of trapping process occurring on the networks under consideration.

III. RANDOM WALKS WITH A FIXED TRAP

In this section, we study the so-called simple discrete-time random walks of a particle on network H_n. At each time step, the particle (walker) jumps from its current location to one of its neighbors with equal probability. In particular, we focus on the trapping problem, i.e., a special issue for random walks with a trap positioned at a given node. To this end, we first distinguish different nodes in H_n by labeling them in the following way. The two nodes in H_0 have labels 1 and 2. For each new generation, we only label the new nodes created at this generation while we keep the labels of all pre-existing nodes unchanged. In other words, we label sequentially new nodes as $M+1, M+2, \dots, M+\Delta M$, where M is the number of the old nodes and ΔM is the number of newly created nodes. In this way, every node is labeled by a unique integer, at generation n all nodes are labeled from 1 to $V_n=\frac{2}{3}(4^n+2)$. Figures 4 and 5 show how the nodes are labeled for two special cases of $q=1$ and $q=0$.

We place the trap at node 1 denoted by i_T. At each time step, the particle, starting from any node except the trap i_T, moves uniformly to any of its nearest neighbors. It should be mentioned that, due to the symmetry, the trap can be also situated at nodes 2, 3, or 4, which has not any effect on MFPT. The special selection we made for the trap allows to address the issue conveniently. Particularly, this makes it possible to analytically compute the MFPT for the two deterministic networks corresponding to $q=1$ and $q=0$ (details will be discussed below) because of their special structures and the convenience of identifying the trap i_T since the first generation.

As mentioned above, one of the most important quantity characterizing such a trapping problem is the FPT defined as the expected time a walker takes, starting from a source node, to first reach the trap node. The significance first originates from the fact that the first encounter properties are relevant to those in a plethora of real situations [35], including transport, disease spreading, target search, and so on. On the other hand, many other quantities can be expressed in terms of FPTs and more information about the dynamics of random walks can be extracted from the analysis of FPTs [36]. Finally, the average of first-passage times, i.e., MFPT, measures the efficiency of the trapping process: the smaller the MFPT, the higher the efficiency and vice versa. In the following, we will determine the exact solutions to MFPT for some limiting cases, as well as the dependence relation of MFPT on the network order.

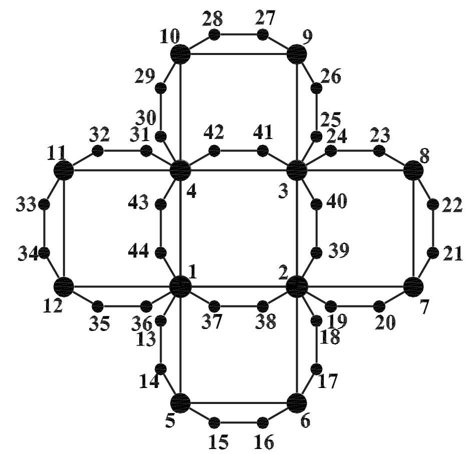

FIG. 4. Labels of all nodes of H_3 in the case of $q=1$.

Let $T_i^{(n)}$ be the FPT for a walker initially placed at node i to first reach the trap i_T in H_n. This quantity can be expressed in terms of mean residence time (MRT) [16,37], which is defined to be the mean time that a random walker spends at a given node prior to being absorbed by the trap. Actually, the MRT is the mean number of visitations of a given node by the walker before trapping occurs.

It is known that the trapping problem studied can be described by a Markov chain [38], whose fundamental matrix is the inverse of matrix $\mathbf{B}_n$ that is a variant of the normalized Laplacian matrix [39] $\mathbf{L}_n$ for H_n and can be obtained from $\mathbf{L}_n$

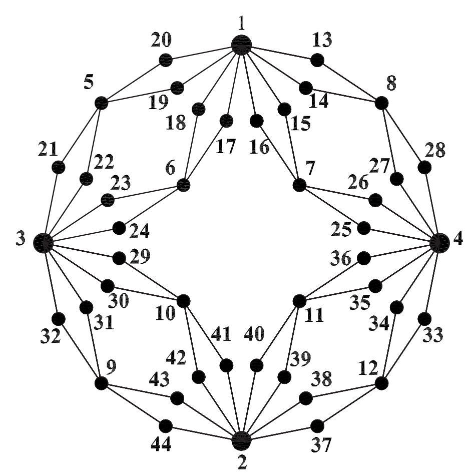

FIG. 5. Labels of all nodes of H_3 for the particular case of $q=0$.

TABLE I. Numerical results of the trapping time $T_i^{(n)}$ for a random walker starting from node i on the network H_n for various n in the case of $q=1$. All the values are obtained through the direct calculation from Eq. (4).

$n\backslash i$	2,3	4	5,6	7,8	9,10	11,12	13,14	15,16	17,18	19,20	21,22	23,24	25,26	27,28	29–32	33–36	37–40	41–44
1	3	4																
2	9	12	5	8	12	13												
3	27	36	15	24	36	39	7	12	20	23	27	28	11	20	32	35	39	40
4	81	108	45	72	108	117	21	36	60	69	81	84	33	60	96	105	117	120
5	243	324	135	216	324	351	63	108	180	207	243	252	99	180	288	315	351	360
6	729	972	405	648	972	1053	189	324	540	621	729	756	297	540	864	945	1053	1080

with all entries in the first row and column (corresponding to the trap node) setting to zeros. The entry $(b_n^{-1})_{ij}$ of the fundamental matrix $(\mathbf{B}_n)^{-1}$ expresses the mean number of visitations of node j by the particle, starting from node i, before it is eventually trapped. Thus, we have

$$T_i^{(n)} = \sum_{j=2}^{V_n} (b_n^{-1})_{ij}. \tag{4}$$

Then, the mean first-passage time, $\langle T\rangle_n$, which is the average of $T_i^{(n)}$ over all initial nodes distributed uniformly over nodes in H_n other than the trap, is given by

$$\langle T\rangle_n = \frac{1}{V_n-1}\sum_{i=2}^{V_n} T_i^{(n)} = \frac{1}{V_n-1}\sum_{i=2}^{V_n}\sum_{j=2}^{V_n} (b_n^{-1})_{ij}. \tag{5}$$

Equation (5) shows that the problem of determining $\langle T\rangle_n$ is reduced to computing the sum of all elements of the fundamental matrix $(\mathbf{B}_n)^{-1}$. Although the expression of Eq. (5) seems compact, the complexity of inverting $\mathbf{L}_n$ is $O(V_n^3)$. Since the network order increases exponentially with n, Eq. (5) becomes intractable for large n. Thus, restricted by time and computer memory, one can obtain $\langle T\rangle_n$ through direct calculation from Eq. (5) only for the first iterations. It would be satisfactory if good alternative computation methods could be proposed to get around this problem. What is encouraging is that the particular construction of the networks and the special choice of the trap location allow to calculate analytically MFPT to obtain a closed-form formula, at least for the two special cases of $q=1$ and $q=0$. The computation details will be provided in the following text.

A. Case of $q=1$

We first establish the scaling relation governing the evolution for $T_i^{(n)}$ with generation n. In Table I, we list the numerical values of $T_i^{(n)}$ for some nodes up to $n=6$. From the numerical values, we can observe that for a given node i, the relation $T_i^{(n+1)}=3T_i^{(n)}$ holds. That is to say, upon growth of the network from generation n to generation $n+1$, the trapping time to first arrive at the trap increases by a factor 3. This is a basic characteristic of random walks on H_n when $q=1$, which can be established from the arguments below [40–42].

Consider an arbitrary node i in H_n of the $q=1$ case after n generation evolution. From Eq. (3), we know that upon growth of the network to generation $n+1$, the degree, k_i, of node i doubles, namely, it increases from k_i to $2k_i$. Among these $2k_i$ neighbors, one half are old neighbors, while the other half are new nodes created at generation $n+1$, each of which has two connections, attached to node i and another simultaneously emerging new node. We now examine the standard random walk in H_{n+1}: let X be the FPT for a particle going from node i to any of its k_i old neighbors; let Y be the FPT for going from any of the k_i new neighbors of i to one of the k_i old neighbors; and let Z represent the FPT for starting from any of new neighbors (added to the network at generation $n+1$) of an old neighbor of i to this old neighbor. Then we can establish the following backward equations:

$$\begin{cases} X = \frac{1}{2} + \frac{1}{2}(1+Y), \\ Y = \frac{1}{2}(1+X) + \frac{1}{2}(1+Z), \\ Z=\frac{1}{2} + \frac{1}{2}(1+Y). \end{cases} \tag{6}$$

Equation (A2) has a solution $X=3$. Thus, upon the growth of the network from generation n to generation $n+1$, the first-passage time from any node i to any node j (both i and j belong to H_n) increases by a factor of 3. That is to say, $T_i^{(n+1)}=3T_i^{(n)}$, which will be useful for the derivation of the exact formula for the MFPT below.

After obtaining the scaling of first-passage time for old nodes, we now derive the analytical rigorous expression for the MFPT $\langle T\rangle_n$. Before proceeding further, we first introduce the notations that will be used in the rest of this section. Let Δ_n denote the set of nodes in H_n, and let $\bar{\Delta}_n$ stand for the set of those nodes entering the network at generation n. For the convenience of computation, we define the following quantities for $1\le m\le n$:

$$T_{m,\mathrm{tot}}^{(n)} = \sum_{i\in\Delta_m} T_i^{(n)} \tag{7}$$

and

$$\bar{T}_{m,\mathrm{tot}}^{(n)} = \sum_{i\in\bar{\Delta}_m} T_i^{(n)}. \tag{8}$$

By definition, it follows that $\Delta_n=\bar{\Delta}_n\cup\Delta_{n-1}$. Thus, we have

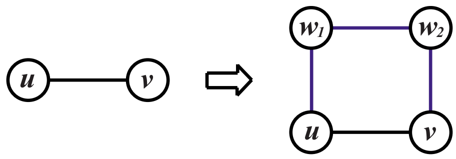

FIG. 6. (Color online) Illustration showing the relation of the mean transmit times for two new nodes and two old nodes connected by an edge generating the new nodes.

$$T_{n,\mathrm{tot}}^{(n)} = T_{n-1,\mathrm{tot}}^{(n)} + \bar{T}_{n,\mathrm{tot}}^{(n)} = 3T_{n-1,\mathrm{tot}}^{(n-1)} + \bar{T}_{n,\mathrm{tot}}^{(n)}, \tag{9}$$

where the relation of $T_i^{(n+1)}=3T_i^{(n)}$ has been made use of. Hence, in order to determine $\bar{T}_{n,\mathrm{tot}}^{(n)}$, we should first find the quantity $\bar{T}_{n,\mathrm{tot}}^{(n)}$ that can be obtained as follows.

By construction, at a given generation, for each edge connecting two nodes u and v (see Fig. 6), it will generate two new nodes (say w_1 and w_2) in the next generation, and the mean transmit times for the two new nodes obey the following relations:

$$\begin{cases} T(w_1) = \frac{1}{2}[1+T(w_2)] + \frac{1}{2}[1+T(u)], \\ T(w_2) = \frac{1}{2}[1+T(w_1)] + \frac{1}{2}[1+T(v)]. \end{cases} \tag{10}$$

In Eq. (10), $T(s)$ represents the expected time of a walker, originating at node s to first get to the trap node. From Eq. (10), we have

$$T(w_1) + T(w_2) = 4 + T(u) + T(v). \tag{11}$$

Summing Eq. (11) over all the E_n old edges pre-existing at the generation n, we obtain

$$\begin{aligned}\bar{T}_{n+1,\mathrm{tot}}^{(n+1)} &= 4E_n + \sum_{i\in\Delta_n} (k_i(n)\times T_i^{(n)}) \\ &= 4^{n+1} + 2\bar{T}_{n,\mathrm{tot}}^{(n+1)} + 2^2\bar{T}_{n-1,\mathrm{tot}}^{(n+1)} + \cdots + 2^n\bar{T}_{1,\mathrm{tot}}^{(n+1)} + 2^n\bar{T}_{0,\mathrm{tot}}^{(n+1)}.\end{aligned} \tag{12}$$

For example, in H_2 (see Fig. 4), $\bar{T}_{2,\mathrm{tot}}^{(2)}$ can be expressed as

$$\begin{aligned}\bar{T}_{2,\mathrm{tot}}^{(2)} &= (T_5^{(2)}+T_6^{(2)}) + (T_7^{(2)}+T_8^{(2)}) + (T_9^{(2)}+T_{10}^{(2)}) + (T_{11}^{(2)}+T_{12}^{(2)}) \\ &= (4+T_1^{(2)}+T_2^{(2)}) + (4+T_2^{(2)}+T_3^{(2)}) + (4+T_3^{(2)}+T_4^{(2)}) \\ &\quad + (4+T_4^{(2)}+T_1^{(2)}) = 4E_1 + 2(T_1^{(2)}+T_2^{(2)}+T_3^{(2)}+T_4^{(2)}) \\ &= 16 + 2\bar{T}_{1,\mathrm{tot}}^{(2)} + 2\bar{T}_{0,\mathrm{tot}}^{(2)}.\end{aligned} \tag{13}$$

Again, for instance, in H_3 (see Fig. 4), $\bar{T}_{3,\mathrm{tot}}^{(3)}$ can be written as

$$\bar{T}_{3,\mathrm{tot}}^{(3)} = 64 + 2\bar{T}_{2,\mathrm{tot}}^{(3)} + 4\bar{T}_{1,\mathrm{tot}}^{(3)} + 4\bar{T}_{0,\mathrm{tot}}^{(3)}. \tag{14}$$

Now, we can determine $\bar{T}_{n,\mathrm{tot}}^{(n)}$ through a recurrence relation, which can be obtained easily. From Eq. (12), it is not difficult to write out $\bar{T}_{n+2,\mathrm{tot}}^{(n+2)}$ as

$$\begin{aligned}\bar{T}_{n+2,\mathrm{tot}}^{(n+2)} &= 4^{n+2} + 2\bar{T}_{n+1,\mathrm{tot}}^{(n+2)} + 2^2\bar{T}_{n,\mathrm{tot}}^{(n+2)} + \cdots + 2^{n+1}\bar{T}_{1,\mathrm{tot}}^{(n+2)} \\ &\quad + 2^{n+1}\bar{T}_{0,\mathrm{tot}}^{(n+2)}.\end{aligned} \tag{15}$$

Equation (15) minus Eq. (12) times six and applying the relation of $T_i^{(n+2)}=3T_i^{(n+1)}$, one gets the following recurrence relation:

$$\bar{T}_{n+2,\mathrm{tot}}^{(n+2)} = 12\bar{T}_{n+1,\mathrm{tot}}^{(n+1)} - 2\times 4^{n+1}. \tag{16}$$

Using $\bar{T}_{1,\mathrm{tot}}^{(1)}=7$, Eq. (16) is solved inductively

$$\bar{T}_{n,\mathrm{tot}}^{(n)} = 4^{n-1} + 6\times 12^{n-1}. \tag{17}$$

Inserting Eq. (17) into Eq. (9) leads to

$$T_{n,\mathrm{tot}}^{(n)} = 3T_{n-1,\mathrm{tot}}^{(n-1)} + 4^{n-1} + 6\times 12^{n-1}. \tag{18}$$

Considering the initial condition $T_{1,\mathrm{tot}}^{(1)}=10$, Eq. (18) is resolved by induction to obtain

$$T_{n,\mathrm{tot}}^{(n)} = \frac{2}{3}\times 12^n + 4^n - 2\times 3^{n-1}. \tag{19}$$

Substituting Eq. (19) into Eq. (5), we obtain the closed-form expression for the MFPT for the trapping problem on H_n of the $q=1$ case as follows:

$$\langle T\rangle_n = \frac{1}{V_n-1}T_{n,\mathrm{tot}}^{(n)} = \frac{1}{2\times 4^n+1}(2\times 12^n + 3\times 4^n - 2\times 3^n). \tag{20}$$

Below we will show how to express $\langle T\rangle_n$ in terms of network order V_n, with the aim of obtaining the relation between these two quantities. Recalling Eq. (1), we have $4^n = \frac{3}{2}V_n - 2$ and $n=\log_4(\frac{3}{2}V_n-2)$. Thus, Eq. (20) can be rewritten as

$$\langle T\rangle_n = \frac{V_n-2}{V_n-1}\left(\frac{3}{2}V_n-2\right)^{\ln 3/\ln 4} + \frac{3V_n-4}{2(V_n-1)}. \tag{21}$$

For large network, i.e., $V_n\to\infty$,

$$\langle T\rangle_n \sim (V_n)^{\ln 3/\ln 4}, \tag{22}$$

with the exponent less than 1. Thus, in large network the MFPT grows sublinearly with network order.

It should be mentioned that both the network corresponding to $q=1$ addressed above and the network discussed in [14] are the particular cases of the $(1,y)$-flower initially introduced in [24]. Then, it is natural to expect that the analytical method for computing the MFPT used above can be extended to applied to calculate the MFPT for trapping on the $(1,y)$-flower. In the Appendix, we show the how to generalize the above technique to determine the MFPT for the $(1,y)$-flower and show how the MFPT changes with network order and parameter y.

B. Case of $q=0$

Analogous to the case of $q=1$, before deriving the general formula for $\langle T\rangle_n$ for the limiting case of $q=0$, we first establish the scaling relation dominating $T_i^{(n)}$ evolving with generation n. To attain this goal, we examine the numerical values of $T_i^{(n)}$ for some nodes up to $n=6$, which can be obtained straightforwardly via Eq. (4). From the numerical results listed in Table II, one can easily observe that for a given node

TABLE II. The trapping time $T_i^{(n)}$ for a random walker starting from node i on the network H_n for various n in the case of $q=0$. All the values are calculated straightforwardly from Eq. (4).

$n\backslash i$	2	3,4	5–8	9–12	13–20	21–28	29–36	37–44
1	4	3						
2	16	12	7	15				
3	64	48	28	60	15	39	55	63
4	256	192	112	240	60	156	220	252
5	1024	768	448	960	240	624	880	1008
6	4096	3072	1792	3840	960	2496	3520	4032

i, its MFPT changes with the generation as $T_i^{(n+1)}=4T_i^{(n)}$, which can be supported by the following argument.

Consider a node i in the nth generation of network H_n for a particular case of $q=0$. In the generation $n+1$, its degree k_i doubles by growing from k_i to $2k_i$. Moreover, different from that of the $q=1$ case, all the $2k_i$ neighbors of node i are new nodes created at generation $n+1$. We now examine the random walks taking place in H_{n+1}: let X be the FPT originating at node i to any of its k_i old neighbors, i.e., those nodes directly connected to i at iteration n; and let Y denote FPT for going from any of the $2k_i$ new neighbors of i to one of its k_i old neighbors. Then the following relations hold:

$$\begin{cases} X = 1 + Y, \\ Y = \frac{1}{2} + \frac{1}{2}(1+X). \end{cases} \tag{23}$$

Equation (23) has a solution $X=4$ found by eliminating Y, which means that for any pair of nodes i and j in H_n, the FPT from i to j increases by a factor of four during the growth of the network from generation n to generation $n+1$. The relation $T_i^{(n+1)}=4T_i^{(n)}$ is a basic feature for random walks on the network of the $q=0$ case, which will be applied to the derivation of the exact formula for $\langle T\rangle_n$.

Having obtaining the evolution relation of trapping time for old nodes when the network grows, we continue to derive the analytical rigorous expression for the MFPT. In what follows, we will use the same notations as those for the $q=1$ case defined above. Similar to the $q=1$ case, it is easy to get the following equation:

$$T_{n,\mathrm{tot}}^{(n)} = T_{n-1,\mathrm{tot}}^{(n)} + \bar{T}_{n,\mathrm{tot}}^{(n)} = 4T_{n-1,\mathrm{tot}}^{(n-1)} + \bar{T}_{n,\mathrm{tot}}^{(n)}. \tag{24}$$

Therefore, to determine $T_{n,\mathrm{tot}}^{(n)}$, we need to find $\bar{T}_{n,\mathrm{tot}}^{(n)}$ first, which can be obtained as follows. Notice that for any given edge attaching two nodes u and v (see Fig. 7) in H_n, it will generate two new nodes (w_1 and w_2) in H_{n+1}, and the FPTs for the two new nodes are equal to each other obeying the following equation:

$$T(w_1) = T(w_2) = 1 + \frac{1}{2}[T(u) + T(v)], \tag{25}$$

which yields to

$$T(w_1) + T(w_2) = 2 + T(u) + T(v). \tag{26}$$

Summing Eq. (26) over all E_n old edges belonging to H_n, we have

$$\begin{aligned} \bar{T}_{n+1,\mathrm{tot}}^{(n+1)} &= 2E_n + \sum_{i\in\Delta_n} (k_i(n) \times T_i^{(n)}) \\ &= 2\times 4^n + 2\bar{T}_{n,\mathrm{tot}}^{(n+1)} + 2^2\bar{T}_{n-1,\mathrm{tot}}^{(n+1)} + \cdots + 2^n\bar{T}_{1,\mathrm{tot}}^{(n+1)} \\ &\quad + 2^n\bar{T}_{0,\mathrm{tot}}^{(n+1)}, \end{aligned} \tag{27}$$

from which we can derive the following recursive relation:

$$\bar{T}_{n+2,\mathrm{tot}}^{(n+2)} = 16\bar{T}_{n+1,\mathrm{tot}}^{(n+1)} - 2\times 4^{n+1}. \tag{28}$$

Using the initial condition $\bar{T}_{1,\mathrm{tot}}^{(1)}=6$, Eq. (28) is solved inductively to get

$$\bar{T}_{n,\mathrm{tot}}^{(n)} = \frac{4^n}{6} + \frac{4^{2n}}{3}. \tag{29}$$

Plugging Eq. (29) into Eq. (24), we have

$$T_{n,\mathrm{tot}}^{(n)} = 4T_{n-1,\mathrm{tot}}^{(n-1)} + \frac{4^n}{6} + \frac{4^{2n}}{3}. \tag{30}$$

Combining with the initial condition $T_{1,\mathrm{tot}}^{(1)}=10$, one can solve Eq. (30) by induction to obtain

$$T_{n,\mathrm{tot}}^{(n)} = \frac{4^n}{18}(8\times 4^n + 3n + 10). \tag{31}$$

Inserting Eq. (31) into Eq. (5), we obtain the rigorous solution for the MFPT for the trapping issue performed on H_n of the $q=0$ case:

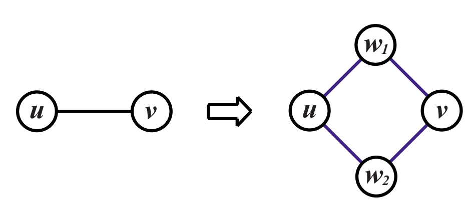

FIG. 7. (Color online) Illustration showing the relation of the first passage times for two new nodes and two old nodes connected at last generation by an edge creating the new nodes.

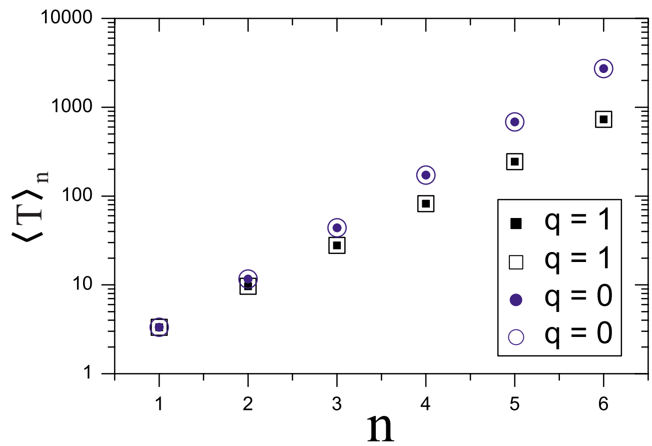

FIG. 8. (Color online) Mean first-passage time $\langle T\rangle_n$ as a function of the iteration n on a semilogarithmic scale for two case of $q=1$ and $q=0$. The filled symbols are the numerical results obtained by direct calculation from Eq. (5), while the empty symbols correspond to the exact values from Eqs. (20) and (32). The analytical and numerical values are consistent with each other.

$$\langle T\rangle_n = \frac{1}{V_n-1}T_{n,\mathrm{tot}}^{(n)} = \frac{4^n}{6(2\times 4^n+1)}(8\times 4^n+3n+10). \tag{32}$$

As in the case of $q=1$, we can recast $\langle T\rangle_n$ as a function of the network order:

$$\langle T\rangle_n = \frac{3V_n-4}{36(V_n-1)}\left[12V_n+\frac{3\ln\left(\frac{3}{2}V_n-2\right)}{2\ln 2}-6\right], \tag{33}$$

from which it is easy to see that for large network (i.e., $V_n \to \infty$), we have the following expression:

$$\langle T\rangle_n \approx V_n. \tag{34}$$

Thus, the MFPT grows linearly with increasing order of network, which is in sharp contrast to the sublinear scaling for the $q=1$ case shown above.

In order to confirm the analytical expressions provided by Eqs. (20) and (32), we have compared the exact solutions for the MFPT with numerical values given by Eq. (5) (see Fig. 8). For all $1\le n\le 6$, the analytical values obtained from Eqs. (20) and (32) show complete agreement with their corresponding numerical results. This agreement is an independent test of our theoretical formulas.

C. Case of $0<q<1$

We have obtained the explicit expressions for MFPT of random walks with a trap on the networks for two limiting cases of $q=1$ and $q=0$ and shown that for the corresponding cases the MFPT grows sublinearly or linearly with the network order. But for the case of $0<q<1$, there are some difficulties in obtaining a closed formula for $\langle T\rangle_n$ as for the two special cases of $q=1$ and $q=0$, since for $q=1$ and $q=0$, the networks are deterministic and self-similar, which allows one derive the analytic solutions for $\langle T\rangle_n$; while for $0<q<1$, the networks are stochastic, which makes it impossible to write a recursive relation for the evolution of the first-passage time.

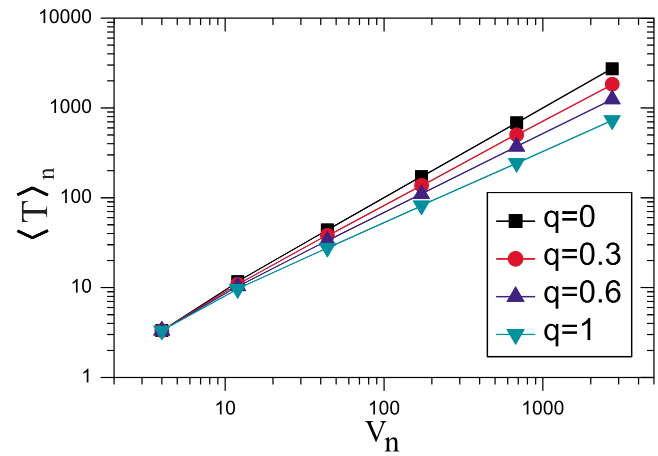

FIG. 9. (Color online) Mean first-passage time $\langle T\rangle_n$ versus the network order V_n on a log-log scale for various n and q. The solid lines serve as guides to the eye.

In order to obtain the dependence relation of MFPT with the network order for $0<q<1$, we have performed extensive numerical simulations for various networks with different iteration n $(1\le n\le 6)$ and q between 0 and 1. Figure 9 illustrates the variation in MFPT with network order V_n, showing that for all $0\le q\le 1$, the MFPT grows as a power-law function of V_n with the exponent $\theta(q)$ changing with q: When q increases from 0 to 1, the exponent $\theta(q)$ decreases from 1 to $\ln 3/\ln 4$.

From Fig. 9 we also know that the efficiency of trapping process is closely related to parameter q: the larger the parameter q, the higher of the efficiency of the trapping problem. To show this concretely, we performed numerical calculation for network H_6 with order 2732 for different q. For each fraction q $(0<q<1)$, all results are obtained by applying Eq. (5) to an ensemble of 100 network realizations. In Fig. 10, we plot the MFPT, $\langle T\rangle_6$, as a function of q. It is easily observed that when q increases from 0 to 1, the MFPT decreases monotonically with q.

The above described phenomenon that the leading behavior of MFPT is a decreasing function of parameter q can be

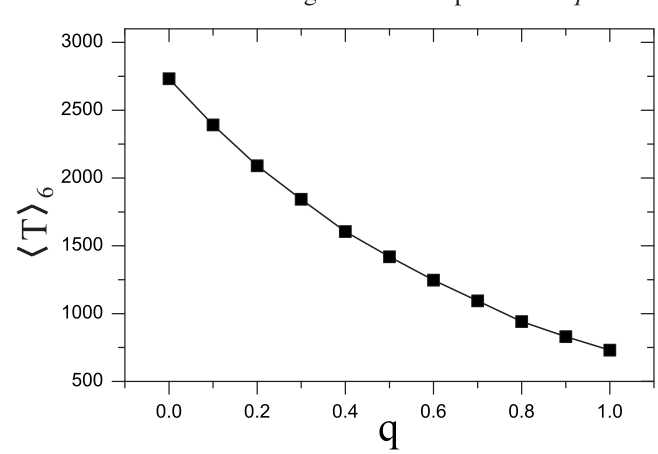

FIG. 10. Dependence relation of MFPT on parameter q.

explained by the following heuristic arguments. For the case of $q=1$, nodes (including the trap) with large degrees are directly linked to one another. In addition, these large-degree nodes are also connected to nodes with small degrees. In this liming case, the network is a very compact system [23,24]. Thus, a walker can easily visit large-degree nodes irrespective of its starting location. The interconnection within large-degree nodes makes the walker spend a short time to hit the trap. In contrast, for $q=0$, the network is fractal, the large-degree nodes are completely repulsive and are exclusively connected to small nodes with a degree of two [23,24,30,31]. When a walker jumps in this network, it will generally first arrive at some large-degree node far from the trap because of the repulsion between the large-degree nodes, and prior to being absorbed, the walker will spend much time in the intermediate region between its starting point and the trap node. Hence, the trapping time is longer than the $q=1$ case. In the case of $0<q<1$, the network interpolates between the case $q=0$ and $q=1$. When q increases from 0 to 1, the extent of repulsion between large nodes becomes less, and thus the APL lessens [23]. Therefore, the MFPT decreases as q increases.

Moreover, since for the two liming cases of $q=0$ and $q=1$, the MFPT $\langle T\rangle_n$ behaves as a power-law function of network order V_n, we thus expect that in the intermediate region $0<q<1$, the MFPT also scales as $\langle T\rangle_n \approx (V_n)^{\theta(q)}$ with $\theta(q)$ being a decreasing function of q as seen from Fig. 9. Although it is difficult to know exactly how does $\theta(q)$ drop with q, Fig. 10 reveals that different q has an obviously distinct effect on the MFPT. For example, when q is in the vicinity of 0, a small increase in q makes the MFPT drop rapidly, while the same change in q near 1 leads to less effect. This phenomenon is analogous to that seen in the Watts-Strogatz (SW) small-world network model [43], where the probability p for rewiring each edge plays a similar role as the parameter q here.

IV. CONCLUSIONS

In summary, we have investigated the trapping issue on a family of scale-free networks with identical degree sequence thus the same degree distribution, which is controlled by a parameter $q(0\le q\le 1)$. We computed analytically or numerically the mean first-passage time (MFPT) for the trapping problem on the networks for various q. The obtained results show that for all q, the MFPT grows as a power-law function of network order with the exponent $\theta(q)$ dependent on q: when the parameter q grows from 0 to 1, the exponent $\theta(q)$ decreases from 1 to $\ln 3/\ln 4$, which indicates that power-law degree distribution alone is not enough to characterize the trapping problem performed on scale-free networks. Therefore, when one makes general statements about the behavior of trapping issue on scale-free networks, care should be needed.

ACKNOWLEDGMENTS

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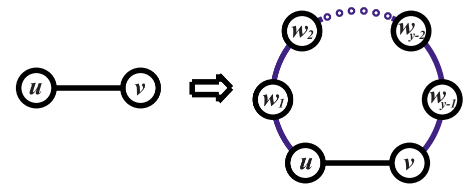

FIG. 11. (Color online) Construction method of the $(1,y)$-flower: each edge connecting two nodes u and v generates $y-1$ new nodes denoted by $w_1, w_2, \ldots, w_{y-1}$, respectively.

APPENDIX: DERIVATION OF MFPT FOR RANDOM WALKS ON (1,y)-FLOWER

The $(1,y)$-flower $(y\ge 2)$ proposed in [24] are constructed in an iterative way. For clarity, we also use H_n to represent the $(1,y)$-flower. Initially $(n=0)$, H_0 is an edge (a link) connecting two nodes. For $n\ge 1$, H_n is obtained from H_{n-1}: for each edge linking two nodes u and v in H_{n-1}, $y-1$ new nodes are added. These $y-1$ nodes, together with u and v, form a path of y links long, see Fig. 11. For the special cases of $y=2$ and $y=3$, the $(1,y)$-flower is reduced to the two networks studied in [14] and Sec. III A, respectively.

It is easy to see that after n iterations, the total number of edges in the $(1,y)$-flower is $E_n=(y+1)^n$. By construction, at each iteration $n_i(n_i\ge 1)$, $L(n_i)=(y-1)E_{n_i}$ new nodes are created. Thus, the number of all nodes in nth generation $(1,y)$-flower is

$$V_n = \sum_{n_i=0}^{n} L(n_i) = \frac{y-1}{y}(y+1)^n + \frac{y+1}{y}. \tag{A1}$$

The $(1,y)$ flower exhibits an obvious feature that the degree of all existing nodes doubles at the next iteration. In network H_n, there exist only nodes of degree $k=2^m(1\le m\le n)$, and the number of nodes having this degree is $L(n-m+1)+2\delta_{m,n}$.

The $(1,y)$-flower has an interesting topology observed in various real networks [24]. First, it is scale free with a power-law degree distribution $P(k)\sim k^{-\gamma}$ with the degree exponent $\gamma=1+\ln(1+y)/\ln 2$. Since $y\ge 2$, $\gamma\in[1+\ln 3/\ln 2,\infty)$. Second, the network H_n is small-world, its diameter, defined as the maximum of the shortest distances over all node pairs, is $(y-1)n$.

Having introducing the $(1,y)$-flower, next we study the trapping issue on it with the immobile trap located at either of the two nodes generated at iteration 0. In the case without any confusion, below we will make use of the same notations applied in the text. We first study the evolution of trapping time $T_i^{(n)}$ of node i with generation n. For this purpose, we give the following definition. At iteration $n+1$, for the $y-1$ new nodes $(v_1, v_2, \ldots, v_{y-1})$ created by an edge incident on an old node i and one of its old neighbors (by construction, these $y+1$ nodes constitute a path of y links long). We call v_1 a first-order new (direct) neighbor of i, v_2 a second-order new neighbor of i, v_3 a third-order new neighbor of i, and so on.

Consider a random walk in H_{n+1}: as shown above, when the network grows from generation n to $n+1$, the degree, k_i, of node i doubles, increasing from k_i to $2k_i$. Among these $2k_i$ neighbors, one half are the old neighbors and the other half are the first-order new neighbors. Notice that the number of its other different order new neighbors are also k_i. Let X be the FPT for a particle going from node i to any of its k_i old neighbors; let Y_x $(1 \le x \le y-1)$ be the FPT for going from any of the k_i xth order new neighbors of i to one of its k_i old neighbors. These various FPTs follow the backward equations:

$$\begin{cases} X = \frac{1}{2} + \frac{1}{2}(1+Y_1), \\ Y_1 = \frac{1}{2}(1+X) + \frac{1}{2}(1+Y_1), \\ Y_2 = \frac{1}{2}(1+Y_1) + \frac{1}{2}(1+Y_3), \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ Y_{y-1} = \frac{1}{2} + \frac{1}{2}(1+Y_{y-2}). \end{cases} \tag{A2}$$

Elimination $Y_1, Y_1, \ldots, Y_{y-1}$ leads to $X=y$. Thus, we have the relation $T_i^{(n+1)} = yT_i^{(n)}$. That is, upon growth of the $(1,y)$-flower from generation n to generation $n+1$, the FPT between i and any one of the old nodes j (i.e., $i \in H_n$ and $j \in H_n$) increases by a factor y. This is a fundamental property of random walks on the $(1,y)$ flower and is very useful for the following derivation.

We proceed to derive the analytical solution to the MFPT $\langle T \rangle_n$. First, we compute $T_{n,\text{tot}}^{(n)}$ that obeys the relation

$$T_{n,\text{tot}}^{(n)} = T_{n-1,\text{tot}}^{(n)} + \bar{T}_{n,\text{tot}}^{(n)} = yT_{n-1,\text{tot}}^{(n-1)} + \bar{T}_{n,\text{tot}}^{(n)}. \tag{A3}$$

Prior to finding $T_{n,\text{tot}}^{(n)}$, one must determine the $\bar{T}_{n,\text{tot}}^{(n)}$ first. According to the construction rule (see Fig. 11), for a given link incident on two nodes u and v in H_n, it will lead to the generation of $y-1$ new nodes $(w_1, w_2, \ldots, w_{y-1})$ at generation $n+1$. The FPTs for these $y-1$ new nodes satisfy equations:

$$\begin{cases} T(w_1) = \frac{1}{2}[1+T(w_2)] + \frac{1}{2}[1+T(u)], \\ T(w_2) = \frac{1}{2}[1+T(w_1)] + \frac{1}{2}[1+T(w_3)], \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ T(w_{y-1}) = \frac{1}{2}[1+T(w_{y-2})] + \frac{1}{2}[1+T(v)]. \end{cases} \tag{A4}$$

Equation (A4) yields

$$T(w_1) + T(w_{y-1}) = 2(y-1) + T(u) + T(v). \tag{A5}$$

Analogously, we can obtain

$$T(w_2) + T(w_{y-2}) = 2(y-3) + T(w_1) + T(w_{y-1}), \tag{A6}$$

$$T(w_3) + T(w_{y-3}) = 2(y-5) + T(w_2) + T(w_{y-2}), \tag{A7}$$

and so forth. From these relations, we have

$$\sum_{x=1}^{y-1} T(w_x) = \frac{y(y^2-1)}{6} + \frac{y-1}{2}[T(u)+T(v)]. \tag{A8}$$

Summing Eq. (A8) over all the E_n old edges pre-existing in H_n yields

$$\begin{aligned} \bar{T}_{n+1,\text{tot}}^{(n+1)} &= \frac{y(y^2-1)}{6}E_n + \sum_{i \in \Delta_n}\left(k_i(n) \times \frac{y-1}{2}T_i^{(n)}\right) \\ &= \frac{y(y-1)(y+1)^{n+1}}{6} + (y-1)\bar{T}_{n,\text{tot}}^{(n+1)} + 2(y-1)\bar{T}_{n-1,\text{tot}}^{(n+1)} \\ &\quad + \cdots + 2^{n-1}(y-1)\bar{T}_{1,\text{tot}}^{(n+1)} + 2^{n-1}(y-1)\bar{T}_{0,\text{tot}}^{(n+1)}. \end{aligned} \tag{A9}$$

In the same way, we have

$$\begin{aligned} \bar{T}_{n+2,\text{tot}}^{(n+2)} &= \frac{y(y-1)(y+1)^{n+2}}{6} + (y-1)\bar{T}_{n+1,\text{tot}}^{(n+2)} + 2(y-1)\bar{T}_{n,\text{tot}}^{(n+2)} \\ &\quad + \cdots + 2^{n}(y-1)\bar{T}_{1,\text{tot}}^{(n+2)} + 2^{n}(y-1)\bar{T}_{0,\text{tot}}^{(n+2)}. \end{aligned} \tag{A10}$$

Equation (A10) minus Eq. (A9) times $2y$ and considering $T_i^{(n+2)} = yT_i^{(n+1)}$, we obtain the recursion relation

$$\bar{T}_{n+2,\text{tot}}^{(n+2)} = y(y+1)\bar{T}_{n+1,\text{tot}}^{(n+1)} - \frac{y(y-1)^2(y+1)^{n+1}}{6}. \tag{A11}$$

Using $\bar{T}_{1,\text{tot}}^{(1)} = \frac{y(y-1)(y+4)}{6}$ allows to solve Eq. (A11) by induction

$$\bar{T}_{n,\text{tot}}^{(n)} = \frac{y-1}{6(y+1)}(y+1)^n[y+(y+3)y^n]. \tag{A12}$$

Substituting Eq. (A12) for $\bar{T}_{n,\text{tot}}^{(n)}$ into Eq. (A3),

$$T_{n,\text{tot}}^{(n)} = yT_{n-1,\text{tot}}^{(n-1)} + \frac{y-1}{6(y+1)}(y+1)^n[y+(y+3)y^n]. \tag{A13}$$

Using $T_{1,\text{tot}}^{(1)} = \frac{y(y+1)(y+2)}{6}$, Eq. (A13) is resolved by induction to obtain

$$\begin{aligned} T_{n,\text{tot}}^{(n)} &= \frac{(y+3)(y-1)}{6y}[y(y+1)]^n + \frac{y(y-1)}{6}(y+1)^n \\ &\quad - \frac{y^3-4y-3}{6}y^{n-1}. \end{aligned} \tag{A14}$$

When $y=3$, Eq. (A14) reduces to Eq. (19) in the text.

From Eq. (A1), we have $(y+1)^n = \frac{y}{y-1}V_n - \frac{y+1}{y-1}$ and $n = \log_{y+1}\left(\frac{y}{y-1}V_n - \frac{y+1}{y-1}\right)$. Thus, Eq. (A14) can be rewritten in terms of network order V_n

$$T_{n,\mathrm{tot}}^{(n)} = \left[\frac{y+3}{6}V_n - \frac{(y+3)(y+1)}{6y}\right]\left(\frac{y}{y-1}V_n - \frac{y+1}{y-1}\right)^{\ln y/\ln(y+1)} + \frac{y(y-1)}{6}\left(\frac{y}{y-1}V_n - \frac{y+1}{y-1}\right) - \frac{y^3-4y-3}{6y}\left(\frac{y}{y-1}V_n - \frac{y+1}{y-1}\right)^{\ln y/\ln(y+1)}. \qquad (A15)$$

Then, by definition, in the large limit of network order, the MFPT $\langle T\rangle_n$ is

$$\langle T\rangle_n = \frac{T_{n,\mathrm{tot}}^{(n)}}{V_n - 1} \sim (V_n)^{\ln y/\ln(y+1)} = (V_n)^{\theta(y)}, \qquad (A16)$$

reproducing the result in Eq. (22) in the case of $y=3$.

Equation (A16) implies that, the leading behavior of MFPT increases as a power-law function of network order. It is not difficult to see that the exponent $\theta(y)=\frac{\ln y}{\ln(y+1)}$ increases with y: when y enhances from 2 to ∞, $\theta(y)$ grows from $\ln 2/\ln 3$ to 1. Equation (A16) also shows that the transportation efficiency declines with parameter y: the larger the parameter y, the less the efficiency. This is easily understood: when y becomes larger, the network is more homogeneous.

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